Volatility and arbitrage

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Joint research with
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October 17, 2016
In a stock market, if there is “adequate volatility”, then there is relative arbitrage. We shall investigate what “adequate volatility” might mean, when there is long-term arbitrage, and when there is arbitrage over arbitrarily short intervals.
The market

Suppose we have a market of stocks $X_1, \ldots, X_n$ represented by positive continuous semimartingales that satisfy

$$d \log X_i(t) = \gamma_i(t) \, dt + \sum_{\nu=1}^{d} \xi_{i\nu}(t) \, dW_{\nu}(t),$$

for $i = 1, \ldots, n$, where $d \geq n$, $(W_1, \ldots, W_d)$ is a $d$-dimensional Brownian motion, and the processes $\gamma_i$ and $\xi_{i\nu}$ are measurable, adapted to the Brownian filtration, and locally integrable or square-integrable. The process $X_i$ represents the total capitalization of the $i$th company. The market weights are

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)},$$

for $i = 1, \ldots, n$. 
Covariance

The \textit{ijth covariance process} $\sigma_{ij}$ is defined for $i, j = 1, \ldots, n$ by

$$
\sigma_{ij}(t) \triangleq \frac{d\langle \log X_i, \log X_j \rangle_t}{dt} = \sum_{\nu=1}^{d} \xi_{i\nu}(t)\xi_{j\nu}(t), \quad \text{a.s.}
$$

If the eigenvalues of the covariance matrix $\sigma(t) = (\sigma_{ij}(t))$ are uniformly bounded away from zero over an interval $[0, T]$, then the market is said to be \textit{strongly nondegenerate} over the interval.
Portfolios

A portfolio \( \pi \) is defined by its weight processes, \( \pi_1, \ldots, \pi_n \), which are bounded, measurable, adapted to the Brownian filtration, and add up to one. The portfolio value process \( Z_\pi \) represents the (positive) value of the portfolio and satisfies

\[
d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) d \log X_i(t) + \gamma^*_\pi(t) dt, \quad \text{a.s.,}
\]

where the excess growth rate process \( \gamma^*_\pi \) is defined by

\[
\gamma^*_\pi(t) \triangleq \frac{1}{2} \left( \sum_{i=1}^{n} \pi_i(t) \sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t) \pi_j(t) \sigma_{ij}(t) \right).
\]

Due to the first equation, \( \gamma^*_\pi \) is effectively observable.
The market portfolio

The *market portfolio* $\mu$ is defined by the market weights $\mu_1, \ldots, \mu_n$, and

$$Z_\mu(t) = X_1(t) + \cdots + X_n(t), \quad \text{a.s.,}$$

with appropriate initial conditions.

The *ijth relative covariance process* $\tau_{ij}$ is defined for $i, j = 1, \ldots, n$ by

$$\tau_{ij}(t) \triangleq \frac{d \langle \log \mu_i, \log \mu_j \rangle_t}{dt} = \frac{d \langle \log(X_i/Z_\mu), \log(X_j/Z_\mu) \rangle_t}{dt}$$

$$= \sigma_{ij}(t) - \sigma_{i\mu}(t) - \sigma_{j\mu}(t) + \sigma_{\mu\mu}(t), \quad \text{a.s.}$$
Diverse markets

A market is *diverse* over the interval $[0, T]$ if there exists a $\delta > 0$ such that for $i = 1, \ldots, n$,

$$\sup_{t \in [0, T]} \mu_i(t) < 1 - \delta, \quad \text{a.s.}$$

**Lemma.** If a market is strongly nondegenerate and diverse over $[0, T]$, then there exists $\varepsilon > 0$ such that for $i = 1, \ldots, n$,

$$\inf_{t \in [0, T]} \tau_{ii}(t) > \varepsilon, \quad \text{a.s.}$$

**Proof.** (F (2002).) Let $x(t) = (\mu_1(t), \ldots, \mu_i(t) - 1, \ldots, \mu_n(t))$, so $\tau_{ii}(t) = x(t)\sigma(t)x^T(t) \geq c\|x(t)\|^2 > c(1 - \mu_i(t))^2 > c\delta^2$, a.s. \qed
Relative arbitrage

For $T > 0$, there is *relative arbitrage* versus the market on $[0, T]$ if there exists a portfolio $\pi$ such that

$$
\mathbb{P}\left[ \frac{Z_\pi(T)}{Z_\mu(T)} \geq \frac{Z_\pi(0)}{Z_\mu(0)} \right] = 1,
$$

$$
\mathbb{P}\left[ \frac{Z_\pi(T)}{Z_\mu(T)} > \frac{Z_\pi(0)}{Z_\mu(0)} \right] > 0.
$$

It is *strong relative arbitrage* if

$$
\mathbb{P}\left[ \frac{Z_\pi(T)}{Z_\mu(T)} > \frac{Z_\pi(0)}{Z_\mu(0)} \right] = 1.
$$

We are interested in conditions under which volatility produces relative arbitrage.
Functionally generated portfolios

Suppose that $S$ is a positive $C^2$ function defined on a neighborhood of the open simplex

$$\Delta^n = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 1, x_i > 0\}.$$  

Then $S$ generates a portfolio $\pi$ such that

$$d \log \left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = d \log S(\mu(t)) + d\Theta(t), \quad \text{a.s.,}$$

for $t \in [0, T]$, where the drift process $\Theta$ is of bounded variation. The weights $\pi_i$ and drift process $\Theta$ are determined by the partial derivatives of $S$ and the covariance matrix of the market. (F (2002).)
Relative variance and relative arbitrage

**Proposition 1.** If there exists an \( \varepsilon > 0 \) and a \( k \in \{1, \ldots, n\} \) such that \( \tau_{kk}(t) > \varepsilon \) for all \( t \in [0, T] \), a.s., then there exists strong relative arbitrage versus the market over \([0, T]\).

**Proof.** (FKK (2005).) For \( p > 1 \), consider the function \( S(x) = x_k^p \), defined for \( x \in \Delta^n \), the unit simplex in \( \mathbb{R}^n \). The function \( S \) generates the portfolio \( \pi \) with weights

\[
\pi_i(t) = \begin{cases} 
  p - (p - 1)\mu_i(t), & \text{for } i = k, \\
  -(p - 1)\mu_i(t), & \text{otherwise},
\end{cases}
\]

and the value process \( Z_{\pi} \) satisfies

\[
d \log \left( \frac{Z_{\pi}(t)}{Z_{\mu}(t)} \right) = d \log \mu_k^p(t) - \frac{p^2 - p}{2} \tau_{kk}(t) \, dt, \quad \text{a.s.}
\]
Relative variance and relative arbitrage

Essentially, the portfolio $\pi$ holds $p$ dollars of $X_k$ and $-(p - 1)$ dollars of the market portfolio. We have

$$\log \left( \frac{Z_\pi(T)}{Z_\mu(T)} \right) - \log \left( \frac{Z_\pi(0)}{Z_\mu(0)} \right)$$

$$= \log \left( \frac{\mu_k^p(T)}{\mu_k^p(0)} \right) - \frac{p^2 - p}{2} \int_0^T \tau_{kk}(t) \, dt$$

$$\leq -p \log \mu_k(0) - \frac{(p^2 - p)\epsilon T}{2}, \quad \text{a.s.}$$

If $p$ is large enough, then $Z_\pi$ will underperform $Z_\mu$, a.s. By shorting $\pi$ and immersing it in a large amount of the market portfolio, we can construct a long-only portfolio that outperforms $Z_\mu$, a.s., over $[0, T]$. \[
\square
\]
The market excess growth rate $\gamma^*_\mu$ measures the average relative volatility available in the market:

$$\gamma^*_\mu(t) = \frac{1}{2} \left( \sum_{i=1}^{n} \mu_i(t) \sigma_{ii}(t) - \sigma_{\mu\mu}(t) \right)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \mu_i(t) \left( \sigma_{ii}(t) - 2\sigma_{i\mu}(t) + \sigma_{\mu\mu}(t) \right)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \mu_i(t) \tau_{ii}(t), \quad \text{a.s.}$$
Cumulative $\gamma^*_\mu$ for the U.S. market
Market entropy

The *entropy* function $S$ is defined by

$$S(x) \triangleq - \sum_{i=1}^{n} x_i \log x_i,$$

for $x \in \Delta^n$. The entropy function satisfies

$$0 \leq S(x) \leq \log n$$

where the value 0 is attained only at the vertices of the simplex, and $\log n$ is attained only when all the $x_i$ are equal to $1/n$. For a constant $c \geq 0$, we define the *generalized entropy function* by

$$S_c(x) \triangleq S(x) + c, \quad \text{for } x \in \Delta^n.$$
Entropy-weighted portfolios

The generalized entropy function $S_c$ generates the portfolio $\pi$ with weights

$$\pi_i(t) = \frac{c - \log \mu_i(t)}{S_c(\mu(t))} \mu_i(t),$$

and the value process $Z_\pi$ of this entropy-weighted portfolio satisfies

$$d \log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = d \log S_c(\mu(t)) + \frac{\gamma^*_\mu(t)}{S_c(\mu(t))} dt, \quad \text{a.s.}$$
Proposition 2. Suppose that in a market defined for $t \geq 0$ there is an $\varepsilon > 0$ such that for all $t$, $\gamma_\mu^*(t) > \varepsilon$, a.s. Then for large enough $T$, there exists strong relative arbitrage versus the market on $[0, T]$.

Proof. For $c > 0$, consider the portfolio $\pi$ generated by $S_c$. Then

$$
\log \left( \frac{Z_\pi(T)}{Z_\mu(T)} \right) - \log \left( \frac{Z_\pi(0)}{Z_\mu(0)} \right) = \log \left( \frac{S_c(\mu(T))}{S_c(\mu(0))} \right) + \int_0^T \frac{\gamma_\mu^*(t)}{S_c(\mu(t))} dt
$$

$$
> \log \left( \frac{c}{c + \log n} \right) + \frac{\varepsilon T}{c + \log n}, \quad \text{a.s.}
$$

Hence, it is just a matter of choosing $T$ large enough. \qed
Short-term relative arbitrage

It would perhaps be nice if \( \gamma^*_\mu(t) > \varepsilon > 0 \) implied short-term relative arbitrage, but this is not quite true. Instead:

**Proposition 3.** For \( T > 0 \), suppose that there exists an \( \varepsilon > 0 \) such that

\[
\gamma^*_\mu(t) > \varepsilon, \quad \text{a.s.,}
\]

for all \( t \in [0, T] \), and that for the entropy function \( S \),

\[
\text{ess inf}\{ S(\mu(t)) : t \in [0, T/2] \} 
\leq \text{ess inf}\{ S(\mu(t)) : t \in [T/2, T] \}.
\]

Then there exists relative arbitrage versus the market on \([0, T]\).
Short-term relative arbitrage

**Proof.** Let

\[ A = \text{ess inf}\{S(\mu(t)) : t \in [0, T/2]\}. \]

Since \( \gamma^*_\mu(t) \geq \varepsilon > 0 \) on \([0, T]\), a.s., not all the \( \mu_i \) can be constantly equal to \( 1/n \), so

\[ 0 \leq A < \log n, \quad \text{a.s.} \]

Hence, we can choose \( \delta > 0 \) such that

\[ A + 2\delta < \log n, \]

and

\[ \mathbb{P}\left[ \inf_{t \in [0, T/2]} S(\mu(t)) < A + \delta \right] > 0. \]
Short-term relative arbitrage

Let us define the stopping time

$$\tau_1 = \inf \{ t \in [0, T/2] : S(\mu(t)) \leq A + \delta \} \wedge T,$$

in which case

$$\mathbb{P} [\tau_1 \leq T/2] > 0.$$ 

We can now define a second stopping time

$$\tau_2 = \inf \{ t \in [\tau_1, T] : S(\mu(t)) = A + 2\delta \} \wedge T,$$

and we have $\tau_1 \leq \tau_2$, a.s.
Short-term relative arbitrage

Now consider the generalized entropy function

\[ S_\delta(x) \triangleq S(x) + \delta, \]

for the same \( \delta > 0 \) as we chose above, so \( S_\delta(x) \geq \delta \). Let \( \pi \) be generated by \( S_\delta \), and we have

\[
\begin{align*}
\log \left( \frac{Z_\pi(\tau_2)}{Z_\mu(\tau_2)} \right) &- \log \left( \frac{Z_\pi(\tau_1)}{Z_\mu(\tau_1)} \right) \\
= \log S_\delta(\mu(\tau_2)) - \log S_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma^*_\mu(t)}{S_\delta(\mu(t))} \, dt, \quad \text{a.s.,}
\end{align*}
\]

for the times \( \tau_1 \) and \( \tau_2 \).
Short-term relative arbitrage

\[ \log n \]

\[ S(\mu(t)) \]

\[ A, A+, A+2, 0, \log n, \delta, \delta, T/2, T, \tau_1, \tau_2 \]

Graph showing the function \( S(\mu(t)) \) with critical points at \( A, A+\delta, A+2\delta \) and \( \tau_1, \tau_2 \) on the horizontal axis from 0 to \( T \).
Suppose that $\tau_1 \leq T/2$, so $\tau_1 < \tau_2$, a.s. There are two cases:

1. If $\tau_2 < T$, then

$$\log S_\delta(\mu(\tau_2)) - \log S_\delta(\mu(\tau_1)) \geq \log(A + 3\delta) - \log(A + 2\delta) > 0, \; \text{a.s.,}$$

and since

$$\int_{\tau_1}^{\tau_2} \frac{\gamma^*_\mu(t)}{S_\delta(\mu(t))} dt > 0, \; \text{a.s.,}$$

we have

$$\log \left( \frac{Z_\pi(\tau_2)}{Z_\mu(\tau_2)} \right) - \log \left( \frac{Z_\pi(\tau_1)}{Z_\mu(\tau_1)} \right) > 0, \; \text{a.s.}$$
2. If $\tau_2 = T$, then

$$A + \delta \leq S_\delta(\mu(t)) < A + 3\delta, \quad \text{a.s.,}$$

for $t \in [\tau_1, T]$, a.s., so

$$\log S_\delta(\mu(\tau_2)) - \log S_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma^*_\mu(t)}{S_\delta(\mu(t))} \, dt$$

$$> \log \frac{A + \delta}{A + 2\delta} + \frac{\varepsilon T}{2(A + 3\delta)}, \quad \text{a.s.}$$

Again there are two cases:
Short-term relative arbitrage

1. If $A = 0$, let
   
   $$\delta = \frac{\varepsilon T}{6 \log 2},$$
   
   in which case,

   $$\log S_\delta(\mu(\tau_2)) - \log S_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma^*_\mu(t)}{S_\delta(\mu(t))} \, dt > \log \frac{A + \delta}{A + 2\delta} + \frac{\varepsilon T}{2(A + 3\delta)} = 0, \quad \text{a.s.},$$

   so

   $$\log \left( \frac{Z_\pi(\tau_2)}{Z_\mu(\tau_2)} \right) - \log \left( \frac{Z_\pi(\tau_1)}{Z_\mu(\tau_1)} \right) > 0, \quad \text{a.s.}$$
Short-term relative arbitrage

2. If $A > 0$, then

$$\lim_{\delta \downarrow 0} \left[ \log \frac{A + \delta}{A + 2\delta} + \frac{\varepsilon T}{2(A + 3\delta)} \right] = \frac{\varepsilon T}{2A} > 0,$$

so for small enough $\delta > 0$

$$\log S_\delta(\mu(\tau_2)) - \log S_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma(\mu(t))}{S_\delta(\mu(t))} dt$$

$$> \log \frac{A + \delta}{A + 2\delta} + \frac{\varepsilon T}{2(A + 3\delta)} > 0, \quad \text{a.s.,}$$

and

$$\log \left( \frac{Z_\pi(\tau_2)}{Z_\mu(\tau_2)} \right) - \log \left( \frac{Z_\pi(\tau_1)}{Z_\mu(\tau_1)} \right) > 0, \quad \text{a.s.}$$
Short-term relative arbitrage

Now consider the portfolio $\eta$ defined by:

1. For $t \in [0, \tau_1)$, $\eta(t) = \mu(t)$, the market portfolio.

2. For $t \in [\tau_1, \tau_2)$, $\eta(t) = \pi(t)$, the portfolio generated by $S_\delta$ with $\delta$ chosen as in the two cases we considered.

3. For $t \in [\tau_2, T]$, $\eta(t) = \mu(t)$. 
Short-term relative arbitrage

If $\tau_1 = T$, then $\eta(t) = \mu(t)$ for all $t \in [0, T]$, so

$$\log \left( \frac{Z_\eta(T)}{Z_\mu(T)} \right) = \log \left( \frac{Z_\eta(0)}{Z_\mu(0)} \right), \quad \text{a.s.}$$

If $\tau_1 \neq T$, then $\tau_1 \leq T/2$ and $\tau_1 < \tau_2$, a.s. By the construction of $\eta$, we have

\[
\log \left( \frac{Z_\eta(T)}{Z_\mu(T)} \right) - \log \left( \frac{Z_\eta(0)}{Z_\mu(0)} \right) \\
= \log \left( \frac{Z_\pi(\tau_2)}{Z_\mu(\tau_2)} \right) - \log \left( \frac{Z_\pi(\tau_1)}{Z_\mu(\tau_1)} \right) \\
> 0, \quad \text{a.s.},
\]

with the inequality following from two the cases we considered.

Since $P[\tau_1 \neq T] > 0$, there exists relative arbitrage on $[0, T]$. \qed
Adequate volatility

**Corollary.** Suppose that $\gamma^*_\mu(t) > \varepsilon > 0$, a.s., for $t \in [0, T]$, and that the market is strongly nondegenerate over that interval. Then there exists relative arbitrage versus the market on $[0, T]$.

**Proof.** There are two cases:

1. If the market is diverse over $[0, T/2]$, then Proposition 1 ensures short-term strong relative arbitrage.

2. If the market is not diverse over $[0, T/2]$, then $A = 0$ in Proposition 3, and short-term relative arbitrage follows.
An example, with variations

Let \( n = 3 \), let \( T > 0 \), and let \( 0 < a < e^{-T/2}/9 \). Suppose that \((W, \theta, B)\) is a 3-dimensional Brownian motion with the usual filtration \( \mathcal{F} \). For \( t \in [0, T] \) and for \( i = 1, 2, 3 \), define

\[
X_i(t) = e^{W(t) - t/2} \left( \frac{1}{3} + \varphi(t) e^{t/2} \cos (\theta(t) + (i - 1)2\pi/3) \right),
\]

where \( \varphi \) is a martingale driven by \( B \) with \( a < \varphi(t) < 3a \). Then the processes \( X_i \) are martingales, and it can be shown that \( \gamma^*_\mu(t) > 3a^2/4 \). Since the price processes in this market are martingales, relative arbitrage does not exist. Since the motions induced by \( W, \theta, \) and \( \varphi \) span \( \mathbb{R}^3 \), the covariance matrix is nonsingular. This market is not strongly nondegenerate.
An example, with variations

We define an $\mathcal{F}$-martingale $\psi$ for $t \in [0, T]$ by

$$
\psi(t) = \int_0^t (a^2 - \psi^2(s)) dB(s),
$$

and we have

$$
-a < \psi(t) < a, \quad \text{a.s.}
$$

Then define $\varphi$ for $t \in [0, T]$ by

$$
\varphi(t) = 2a + \psi(t),
$$

so

$$
a < \varphi(t) < 3a, \quad \text{a.s.},
$$

and

$$
d\langle \varphi \rangle_t = d\langle \psi \rangle_t = (a^2 - \psi^2(t))^2 dt, \quad \text{a.s.}
$$
An example
Let \( n = 3 \), let \( T > 0 \), and let \( 0 < a < e^{-T/2}/9 \). Suppose that \((W, \theta, B)\) is a 3-dimensional Brownian motion with the usual filtration \( \mathcal{F} \). For \( t \in [0, T] \) and for \( i = 1, 2, 3 \), define

\[
X_i(t) = e^{W(t) - t/2} \left( \frac{1}{3} + \varphi(t)e^{t/2} \cos(\theta(t) + (i - 1)2\pi/3) \right),
\]

where \( \varphi \) is a martingale driven by \( B \) with \( a < \varphi(t) < 3a \). Then the processes \( X_i \) are martingales, and it can be shown that \( \gamma^*_\mu(t) > 3a^2/4 \). Since the price processes in this market are martingales, relative arbitrage does not exist. Since the motions induced by \( W, \theta, \) and \( \varphi \) span \( \mathbb{R}^3 \), the covariance matrix is nonsingular. This market is not strongly nondegenerate.
Variations

Let $n = 3$, let $T > 0$, and let $0 < a < e^{-T/2}/9$. Suppose that $(W, \theta, B)$ is a 3-dimensional Brownian motion with the usual filtration $\mathcal{F}$. For $t \in [0, T]$ and for $i = 1, 2, 3$, define

$$X_i(t) = e^{W(t) - t/2} \left( \frac{1}{3} + ae^{t/2} \cos (\theta(t) + (i - 1)2\pi/3) \right),$$

where $\varphi$ is a martingale driven by $B$ with $a < \varphi(t) < 3a$. Then the processes $X_i$ are martingales, and it can be shown that $\gamma^*_\mu(t) > 3a^2/4$. Since the price processes in this market are martingales, relative arbitrage does not exist. Since the motions induced by $W, \theta, \varphi$ span $\mathbb{R}^3$, the covariance matrix is nonsingular. This market is not strongly nondegenerate.
Variations

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\[
X_i(t) = e^{W(t)-t/2} \left( \frac{1}{3} + a \cos (\theta(t) + (i - 1)2\pi/3) \right),
\]

where \( \varphi \) is a martingale driven by \( B \) with \( a < \varphi(t) < 3a \). Then the processes \( X_i \) are martingales, and it can be shown that \( \gamma^*_\mu(t) > 3a^2/4 \). Since the price processes in this market are martingales, relative arbitrage does not exist. Since the motions induced by \( W, \theta, \) and \( \varphi \) span \( \mathbb{R}^3 \), the covariance matrix is nonsingular. This market is not strongly nondegenerate.
Variations

The weights $\mu_i(t)$ for the model

$$X_i(t) = e^{W(t) - t/2} \left( \frac{1}{3} + a \cos (\theta(t) + (i - 1)2\pi/3) \right),$$

lie in a circle on the simplex $\Delta^3$ centered at $(1/3, 1/3, 1/3)$, so

$$S(\mu(t)) = (\mu_1^2(t) + \mu_2^2(t) + \mu_3^2(t))^{1/2} = \text{const.}$$

$S$ generates a portfolio $\pi$ with value function $Z_\pi$ such that

$$d \log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = d \log S(\mu(t)) - \gamma^*_\pi(t) dt$$

$$= -\gamma^*_\pi(t) dt, \quad \text{a.s.}$$

Since $\gamma^*_\pi(t) > 0$, this produces immediate relative arbitrage.
Let \( n = 3 \), let \( T > 0 \), and let \( 0 < a < e^{-T/2}/9 \). Suppose that \((W, \theta, B)\) is a 3-dimensional Brownian motion with the usual filtration \( \mathcal{F} \). For \( t \in [0, T] \) and for \( i = 1, 2, 3 \), define

\[
X_i(t) = e^{W(t) - t/2} \left( \frac{1}{3} + \varphi(t) \cos \left( \theta(t) + (i - 1)2\pi/3 \right) \right),
\]

where \( \varphi \) is a martingale driven by \( B \) with \( a < \varphi(t) < 3a \). Then the processes \( X_i \) are martingales, and it can be shown that \( \gamma^{\ast}_\mu(t) > 3a^2/4 \). Since the price processes in this market are martingales, relative arbitrage does not exist. Since the motions induced by \( W, \theta, \) and \( \varphi \) span \( \mathbb{R}^3 \), the covariance matrix is nonsingular. This market is not strongly nondegenerate.
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where $\varphi$ is a martingale driven by $B$ with $a < \varphi(t) < 3a$. Then the processes $X_i$ are martingales, and it can be shown that $\gamma^*_\mu(t) > 3a^2/4$. Since the price processes in this market are martingales, relative arbitrage does not exist. Since the motions induced by $W$, $\theta$, and $\varphi$ span $\mathbb{R}^3$, the covariance matrix is nonsingular. This market is not strongly nondegenerate.
Let $n = 3$, let $T > 0$, and let $0 < a < e^{-T/2}/9$. Suppose that $(W, \theta, B)$ is a 3-dimensional Brownian motion with the usual filtration $\mathcal{F}$. For $t \in [0, T]$ and for $i = 1, 2, 3$, define

$$X_i(t) = \left( \frac{1}{3} + \varphi(t)e^{t/2} \cos (\theta(t) + (i - 1)2\pi/3) \right),$$

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Let $n = 3$, let $T > 0$, and let $0 < a < e^{-T/2}/9$. Suppose that $(W, \theta, B)$ is a 3-dimensional Brownian motion with the usual filtration $\mathcal{F}$. For $t \in [0, T]$ and for $i = 1, 2, 3$, define

$$X_i(t) = \kappa(t)\left(\frac{1}{3} + \varphi(t)e^{t/2} \cos(\theta(t) + (i - 1)2\pi/3)\right),$$

where $\varphi$ is a martingale driven by $B$ with $a < \varphi(t) < 3a$. Then the processes $X_i$ are martingales, and it can be shown that $\gamma^*_\mu(t) > 3a^2/4$. Since the price processes in this market are martingales, relative arbitrage does not exist. Since the motions induced by $\kappa > 0$, $\theta$, and $\varphi$ span $\mathbb{R}^3$, the covariance matrix is nonsingular. This market is not strongly nondegenerate.
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**Conclusion:** $\gamma_{\mu}^*(t) > \varepsilon > 0$ will generate relative arbitrage, but not over arbitrarily short intervals.
References

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Thank you!